

Three dimensional Lifshitz black hole and the Korteweg-de Vries equation

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We consider a solution of three dimensional New Massive Gravity with a negative cosmological constant and use the AdS/CTF correspondence to inquire about the equivalent two dimensional model at the boundary. We conclude that there should be a close relation with the Korteweg-de Vries equation.

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Recently, alternative theories of gravity have been conceived in the context of quantum gravity, in which higher order curvature corrections appear naturally. In such a domain, it may be even conceivable that Lorentz invariance is not explicitly realized, though being recovered at a later stage. This is the vein chosen by Horava [1], to introduce a (renormalizable) gravitational theory with a space time anisotropy, but with a flow to general relativity at long distances. A theory of massive gravity in three dimensions has also been considered [2, 3], displaying solutions with the same kind of symmetry as the anisotropic space time considered by Horava but in a relativistic setup.

On the other hand, the AdS/CFT relation percolated beyond the string theory where it has been originally formulated and important developments related AdS gravity to models of condensed matter physics [4]. Special attention has been driven by the question of superconductivity and superfluidity [5] using this approach.

One knows that the Korteweg-de Vries (KdV) equation describes a wide class of physical systems [6]. Its studies have applications that spread from hydrodynamics to condensed matter systems. The KdV equation is very useful when one takes in account both nonlinear and dispersive effects and it was firstly developed to study the evolution of solitary waves in shallow water. Some of its remarkable characteristics are the existence of exact solutions called *solitons* and its interpretation as a completely integrable Hamiltonian system.

In this work we establish a correspondence between a Lifshitz black hole and a classical KdV in the spirit of AdS/CFT correspondence. From the gravity point of view, we start with a black hole solution in three dimensions exhibiting Lifshitz scaling. It has been found in the context of the New Massive Gravity (NMG) [3] by Beato *et al* [7]. At the linearized level the theory is equivalent to the unitary Pauli-Fierz theory [2] for free massive spin-2 gravitons in three dimensions. Here we consider NMG

with a negative cosmological constant, whose action is given by

$$\mathcal{S}_{NMG} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[R - 2\lambda - \frac{1}{m^2} \mathcal{K} \right] \quad , \quad (1)$$

where λ is the three-dimensional cosmological constant and $\mathcal{K} = R_{ab}R^{ab} - \frac{3}{8}R^2$. The associated Euler-Lagrange equation is

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{1}{2m^2}\mathcal{K}_{ab} - \lambda g_{ab} \quad , \quad (2)$$

where the higher-derivative features of the theory are encoded in the tensor \mathcal{K}_{ab} (see reference [3]). The black hole obtained by Beato *et al* is an exact solution of Eq.(2) described by the line element

$$ds^2 = -\frac{r^6}{l^6} f(r) dt^2 + \frac{l^2}{r^2} f(r)^{-1} dr^2 + r^2 d\phi^2 \quad , \quad (3)$$

where $f(r) = \left(1 - \frac{r_+^2}{r^2}\right)$ and the event horizon is denoted by r_+ . This solution preserves the Lifshitz scale symmetry $t \rightarrow \lambda^3 t$, $\phi \rightarrow \lambda x$, $r \rightarrow \lambda^{-1} r$ with $r_+ \rightarrow \lambda^{-1} r_+$.

At the spatial infinity the metric (3) reduces to

$$ds_{r \rightarrow \infty}^2 \rightarrow -\frac{r^6}{l^6} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2 \quad . \quad (4)$$

This asymptotic form of the metric is invariant under temporal and spacial translation and anisotropic dilatation, $t' = (1 + 3\epsilon)t$, $r' = (1 + \epsilon)r$, $\phi' = (1 + \epsilon)\phi$, with ϵ an infinitesimal constant. This is a subgroup of the Lifshitz Group $L(3)$ in three dimensions [8]

$$P_\phi = -i\partial_\phi, \quad H = -i\partial_t, \quad D = -i(3t\partial_t + \phi\partial_\phi + r\partial_r) \quad .$$

The propagation of a massive scalar field $\Psi(t, y, \phi)$ on the background metric (4) at the spatial infinity ($y \equiv r_+/r \rightarrow 0$) is given by

$$\frac{d^2\Psi}{dy^2} - \frac{3}{y} \frac{d\Psi}{dy} - \frac{l^2\mu^2}{y^2} \Psi = 0 \quad , \quad (5)$$

with μ denoting the scalar field mass. The solution near the boundary takes the form

$$\Psi = \phi_1(t, \phi)y^{\Delta_+} + \phi_2(t, \phi)y^{\Delta_-}, \quad (6)$$

with $\Delta_{\pm} = 2 \pm \sqrt{4 + \mu^2 l^2}$. We set $\phi_2(t, \phi) = 0$ in order to have a finite value for Ψ at the boundary, implying that the radial dimension of Ψ is $[\Psi] = \Delta_+$. The result is similar to the corresponding problem in the holographic superconductor as given in [9].

As in the usual search for a conformal field theory on the AdS boundary, we look for a field theory living at $y = 0$, whose fields preserve the same symmetries as the bulk scalar Ψ and its radial dimension Δ_+ . We claim that the best candidate is the integrable Korteweg-de Vries theory (KdV) in (1+1)-dimensions. Letting $\Phi(t, \phi)$ be the fundamental field at the boundary with spatial dimension Δ_+ , the KdV theory is invariant under the Lifshitz transformations and $\Phi \rightarrow \Phi' = (1 - \Delta_+ \epsilon) \Phi$. We thus get a generalization of the KdV equation

$$\partial_t \Phi + \partial_\phi^3 \Phi + \Phi^A \partial_\phi \Phi = 0, \quad (7)$$

where $A = \frac{2}{\Delta_+}$ is a constant determined imposing invariance of the theory under the subgroup $L(3)$, in which

case the bulk dimension is $[\Phi] = \Delta_+$.

Using only symmetry arguments we claim that the holographic field theory defined at the boundary $y = 0$ obeys the equation of motion (7).

The first two terms of (7) arise naturally, while the third one is mandatory in view of renormalization effects for the given conformal dimension. For $\Delta_+ = 2$ we recover the usual KdV theory. In this case the mass of scalar field Ψ propagating in the bulk is precisely $\mu_{LF}^2 = -\frac{4}{l^2}$. An interesting point, is that such mass limit is smaller than the Breitenlohner-Freedman bound [10]. One knows [4] that in three dimensions the BF bound results $\mu_{BF}^2 = -\frac{1}{l^2}$, thus the Lifshitz bound is less restrictive than BF bound, *i.e.*, $\mu_{LF}^2 \leq \mu^2 < \mu_{BF}^2$. This new LF bound can have relevant implications in the holographic description of models in condensed matter.

The Klein-Gordon equation in the background geometry of Lifshitz black hole (3) permits considering the behaviour of the field Φ as a function of the Hawking temperature of the black hole as well as the scaling properties of the observables in the bulk as functions of T . In terms of the event horizon parameter, the temperature is given by $T = r_+^3 / 2\pi l^4$. We separate the Klein-Gordon equation with the Ansatz $\Psi(t, y, \phi) = e^{-i\omega t + ik\phi} Z(y)$ and defining $f(y) = 1 - y^2$. Making the redefinitions $y \rightarrow l^{1/3} \tilde{y}$, $k \rightarrow l^{1/3} \tilde{k}$, we get

$$Z(\tilde{y})'' + \frac{\tilde{y}^2 - 3}{\tilde{y}f(\tilde{y})} Z(\tilde{y})' + \left[\left(\frac{\omega}{2\pi T} \right)^2 \frac{\tilde{y}^4}{f(\tilde{y})^2} - \frac{l^2 \mu^2}{f(\tilde{y})} - \left(\frac{\tilde{k}}{(2\pi T)^{1/3}} \right)^2 \right] Z(\tilde{y}) = 0 \quad . \quad (8)$$

From this equation we see that ω scales linearly with the temperature $\omega \sim T$, while the momentum in the ϕ direction scales as $k \sim T^{1/3}$, which is compatible with what we would expect from a holographic theory with the Lifshitz scaling $t \rightarrow \lambda^3 t$, $y \rightarrow \lambda^{-1} y$ and $\phi \rightarrow \lambda \phi$. Given such temperature scaling in the bulk, we ask how the fields at the boundary scale with the black hole temperature. From the previous section, we know that $\Phi \sim \lambda^{\Delta_+}$ at the boundary, where λ is the same scaling factor which appears in the Lifshitz transformation. In terms of the temperature $\partial_t \sim T$, $\partial_\phi \sim T^{1/3}$, which implies that the fields at the boundary have the following dependence with the black hole temperature $\Phi \sim T^{\frac{\Delta_+}{3}}$.

Already briefly mentioned is the fact that in terms of the scaling properties we see that, for an interacting theory, the last term is compatible with all symmetries, thus, in view of renormalization effects it becomes mandatory.

Furthermore, we address the issue of the scalar quasinormal modes of the three dimensional Lifshitz black hole (3) in the hydrodynamic limit. Quasinormal modes are found by solving the wave equation for matter or gravity,

under the boundary conditions of ingoing waves at horizon and outgoing flux at the infinity for asymptotic flat spacetimes and Dirichlet for AdS-like spacetimes. Under such conditions, the wave equation admits solutions only for a set of discrete complex frequencies called quasinormal frequencies. For a recent review see [11].

It is believed that in general any interacting theory can be described by hydrodynamics in the limit of small frequencies ω and wavenumbers k compared to the temperature T of the system [12]. Defining the quantities $\mathbf{m} = \omega / 2\pi T$, $\mathbf{q} = k / (2\pi T)^{1/3}$ and taking the limit $\mathbf{q} \rightarrow 0$ and $\mu = 0$, the equation of scalar field (8) reads

$$Z''(y) + \frac{(y^2 - 3)}{y(1 - y^2)} Z'(y) - \frac{\mathbf{m}^2 y^4}{(1 - y^2)^2} Z(y) = 0 \quad . \quad (9)$$

In the hydrodynamic limit $\mathbf{m} \ll 1$ we can expand the field $Z(y)$ in power series of \mathbf{m} and \mathbf{q} . As we are interested in the limit $\mathbf{q} = 0$, we can write

$$Z(y) = (1 - y^2)^{-\frac{i\mathbf{m}}{2}} [F_0(y) + i\mathbf{m}F_1(y) + \mathcal{O}(\mathbf{m}^2)] \quad . \quad (10)$$

Putting back that expansion in (9) and solving the differential equations for F_0 and F_1 imposing causal boundary

conditions, the solution to Eq.(9) is

$$Z(y) = A(1 - y^2)^{-\frac{i\mathfrak{m}}{2}} \left[1 + \frac{i\mathfrak{m}}{2}(1 - y^2) + \mathcal{O}(\mathfrak{m}^2) \right], \quad (11)$$

where A is a normalization constant. The Dirichlet boundary condition at spatial infinity $Z(0) = 0$, implies the relation $\omega = (4\pi T)i$, showing that the scalar frequencies in the hydrodynamic limit are purely imaginary. Following the AdS/CFT prescription, a black hole perturbation in the bulk spacetime is equivalent to perturb an approximately thermal state defined at the spacetime boundary $y = 0$. In the view of linear response theory [13], there is a timescale of the thermal dual system which its perturbed state spends to return to thermal equilibrium. In particular, such a thermalization timescale has an interpretation in terms of AdS/CFT correspondence, namely that the characteristic damping time of the fundamental quasinormal frequency of the black hole spacetime is related to the thermalization timescale of the dual field theory at the boundary [14]. Applying such interpretation to our result, we found that the thermalization timescale is given by $\tau = 1/4\pi T$. At high temperatures (compared to the wavenumber k and the frequency ω), we see that the field theory timescale is very small, indicating that a perturbation in the two-dimensional dual thermal field theory is not long-lived, it goes exponentially to zero. It is in accordance to the result that the scalar quasinormal frequency that we found in the bulk is purely imaginary, showing that such perturbation has no considerable oscillation stage.

Furthermore, the evidence that in some regime we have a KdV-like theory at the boundary, as we discussed in the precedent section, is favored since one of the most

interesting features of solitonic solutions is that its equilibrium is stable, at least in a weakly dispersing medium [15]. It means that it is very difficult take the theory out off the equilibrium.

Another quantity that can be calculated using the AdS/CFT prescription is the two-point correlation function associated to the massive scalar field $Z(y)$ in the bulk. Following the Son and Starinets recipe [16, 17] the momentum-space two-point function is given by

$$\langle \mathcal{O}_Z(k, \omega) \mathcal{O}_Z(-k, -\omega) \rangle = -2 \lim_{\epsilon \rightarrow 0} F(k, \omega, \epsilon), \quad (12)$$

where ϵ is the cutoff near the boundary and $F(k, \omega, \epsilon)$ is the flux factor, which in terms of bulk metric and boundary-bulk propagator $\tilde{G}(k, \omega, y)$ read as

$$F(k, \omega, \epsilon) = \lim_{y \rightarrow \epsilon} \sqrt{-g} g^{yy} \tilde{G}(-k, -\omega, y) \partial_y \tilde{G}(k, \omega, y).$$

The boundary-bulk propagator $\tilde{G}(k, \omega, y)$ is solution of Klein-Gordon equation (8) with $\tilde{Z}(k, \omega, y) = \tilde{G}(k, \omega, y) \tilde{Z}(k, \omega, \epsilon)$, where $\tilde{Z}(k, \omega, y)$ is the massive scalar field in the Fourier space and $\tilde{Z}(k, \omega, \epsilon)$ is the value of the field at boundary cutoff $y = \epsilon$. Together with Eq. (8) we impose the boundary conditions: $\tilde{G}(k, \omega, \epsilon) = 1$ and $\tilde{G}(k, \omega, 1)$ being finite. Explicitly, the solution is

$$\tilde{G}(k, \omega, y) = \left(\frac{y}{\epsilon} \right)^{\Delta_+} \left[\frac{y^2 - 1}{\epsilon^2 - 1} \right]^{\frac{b}{2}} \frac{H \left[0, \alpha, b, -\frac{b^2}{4}, c, y^2 \right]}{H \left[0, \alpha, b, -\frac{b^2}{4}, c, \epsilon^2 \right]},$$

where H stands for the confluent Heun function and $\alpha = \sqrt{4 + \mu^2 l^2}$, $b = i l^4 \omega / r_+^3$, $c = \alpha^2 / 4 + l^2 k^2 / 4 r_+^2$. Taking the expansion of $\tilde{G}(k, \omega, y)$ and $\tilde{G}(-k, -\omega, y)$ near $y = \epsilon$ and putting back in Eq.(12) we get

$$\langle \mathcal{O}_Z(k, \omega) \mathcal{O}_Z(-k, -\omega) \rangle = \frac{2r_+^4}{l^4} \lim_{\epsilon \rightarrow 0} \left\{ \frac{y^{2\Delta_+} \epsilon^{-2\Delta_+} \bar{H}[\dots, y^2]}{\bar{H}[\dots, \epsilon^2] H[\dots, \epsilon^2]} \left[\frac{\Delta_+}{y} (y^2 - 1) + \frac{b}{y^2} \right] [H[\dots, y^2] + \partial_y H[\dots, y^2]] \right\},$$

where \bar{H} refers to the confluent Heun function with $b \rightarrow -b$. Expanding the Heun's function up to sixth order, it is straightforward to see that this results only in contact terms in the expression for the two-point correlation function, without any term which could give rise to correlations between points with both spatial and temporal separation. Such localized terms in the correlator expression confirm our claim that the boundary theory is solitonic.

Consider again the scalar field in the Lifshitz black hole $\Psi(t, y, \varphi) = q^{1-\frac{\alpha}{2}}(q-1)^{\frac{b}{2}} X(q) e^{-i\omega t + ik\varphi}$. We get the confluent Heun equation

$$X'' + \left(\frac{1-\alpha}{q} + \frac{1+b}{q-1} \right) X' - \frac{b^2}{4(q-1)} X = 0 \quad (13)$$

where α, b are given above and $q = y^2$. From (13) it is possible to get a Hamiltonian structure [18] in order to get the confluent Painlevé P^{VI} equation [19]

$$q_{tt} + \frac{1}{2} \left[\frac{1}{q} + \frac{1}{q-1} \right] q_t^2 + \frac{1}{t} \left[\frac{(1-\alpha)(q-1)}{q} + 1 \right] q_t + \frac{1}{2t^2} Q_1 - \frac{1}{t} Q_2 - \frac{1}{2} q(q-1)(2q-1) = 0, \quad (14)$$

as the equation of motion with

$$Q_1 = \left[\frac{q(1+b)^2}{(q-1)} - \frac{(q-1)(1-\alpha)^2}{q} \right] \text{ and}$$

$$Q_2 = q(q-1) \left[1 + \frac{b(b-2)}{2} - 2 + \alpha \right].$$

The ARS conjecture [20] states that reductions of solitonic equations is a Painlevé equation. In particular, the reduction of the KdV equation is the Painlevé P^{II} [21]. Equation (14) contains the five remaining Painlevé equations [19]. Thus, we can derive from it the Painlevé P^{II} in order to put the KdV equation in the bulk of the Lifshitz black hole studies. Additionally this realization permits us speculate whether a possible interpretation of the quantum theory living on the boundary could be an Ising model in 2 dimensions. Indeed, the results obtained by McCoy *et.al* [24] and Ablowitz [21] establish a connection between KdV equation and Ising model through the Painlevé equation. Therefore, a general relationship between Lifshitz black hole, KdV equation and Ising model could, in principle, be realized.

(*Concluding remarks*). In this work we have shown through different arguments and results a close relation between a 3-dimensional Lifshitz black hole and a classical KdV theory. Another result that corroborates this interpretation is presented by Troncoso [25].

We believe that the mass bound has implications in all holographic models in condensed matter. In principle, the holographic superconductors can condensate more easily in asymptotically Lifshitz models than asymptotically AdS models.

A question not answered up to now in the context of the AdS/CFT relation is the role played by integrable two dimensional theories. As a matter of fact, many can be seen as a perturbation of conformally invariant models. At the classical level, some are indeed conformally invariant but integrable at the quantum level [22], displaying also quantum conformally invariant solutions, as is the case of the chiral Gross-Neveu models in two dimensions [23]. The KdV equation is, to our knowledge, the first case where the relation can be established. A bulk gravitational theory in asymptotically Lifshitz space equivalent to further integrable models might shed new light in both the structure of integrability as well as in the Gauge/Gravity duality itself.

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